A new proof of Pigozzi theorem

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Bern, July 8th 2008

Outline

Quasivarieties

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Fact

A Class is a quasivariety iff it is definable by quasi-identities, that is universal formulae of the form

$$(\forall \bar{x})\Big[\Big[\bigwedge_{i \leqslant n} p_i(\bar{x}) \approx q_i(\bar{x})\Big] \rightarrow p(\bar{x}) \approx q(\bar{x})\Big].$$

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A quasivariety is finitely based provided it is definable by finitely many quasi-identities.

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Theorem (A. I. Mal'cev, S. Burris)

Each algebra in a quasivariety $\mathcal{R}$ is isomorphic to a subdirect product of $\mathcal{R}$-subdirectly irreducible algebras.

Pigozzi theorem

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$$\mathcal{R} \models (\forall x, u, v)[\Gamma(u, v, x, x) \to u \approx v].$$

2. A quasivariety $\mathcal{R}$ has definable relative principal congruences if there exists an $\mathcal{R}$-congruence formula Γ such that

$$\theta_{\mathcal{R}}(a, b) = \{(c, d) \in A^2 \mid A \models \Gamma(c, d, a, b)\}$$

for all $a, b \in A \in \mathcal{R}$.

Definable $\mathcal{R}$-congruences, continued

Theorem (J. Czelakowski, W. Dziobiak)

The quasivariety $\mathcal{R}$ with definable relative principal congruences is finitely based iff the class $\mathcal{R}_{SI}$ of $\mathcal{R}$-subdirectly irreducible algebras is strictly elementary.

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Let $\Gamma(x, y, u, v)$ define principal congruences in $\mathcal{R}$.

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Proof of if direction.

Let $\Gamma(x, y, u, v)$ define principal congruences in $\mathcal{R}$.

1. There exists a finite set of quasi-identities Σ such that $\mathcal{R} = \mathrm{Mod}(\Sigma) \cap \mathbf{H}(\mathcal{R})$ and $\mathcal{R}_{SI} = \mathrm{Mod}(\Sigma)_{SI} \cap \mathcal{R}$.

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2. There is a formula $\Delta(u, v)$ such that $A \models \Delta(c, d)$ iff

$$\{(e, f) \in A^2 \mid A \models \Gamma(e, f, c, d)\}$$

is a $\mathrm{Mod}(\Sigma)$-congruence of A containing (c, d).

Definable $\mathcal{R}$-congruences, continued

Proof continued.

3. Because $\mathcal{R} \models (\forall u, v)\Delta(u, v)$, there is I a finite set of identities such that

$$\mathcal{R} \models I \quad \text{and} \quad I \cup \Sigma \models (\forall u, v)\Delta(u, v).$$

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4. Let

$$\psi = (\exists u, v)\Big[u \not\approx v \wedge (\forall x, y)\big[x \not\approx y \rightarrow \Gamma(u, v, x, y)\big]\Big]$$

and $\mathcal{R}_{SI} = \mathrm{Mod}(\chi)$.

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5. We have $R_{SI} = \mathrm{Mod}(\Sigma \cup I \cup J)_{SI}$ and thus $R = \mathrm{Mod}(\Sigma \cup I \cup J)$.

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Definition

A quasivariety $\mathcal{R}$ has definable relative principal subcongruences if there are $\mathcal{R}$-congruence formulas Γ_1, Γ_2 such that for all $A \in \mathcal{R}$ and each pair of distinct elements $a, b \in A$, there is a pair of distinct elements $c, d \in A$ such that

$$A \models \Gamma_1(c, d, a, b) \quad \text{and} \quad \theta_{\mathcal{R}}(c, d) = \{(e, f) \mid A \models \Gamma_2(e, f, c, d)\}.$$

Proof of Pigozzi theorem

Fact (Mostly due to K. Baker and J. Wang)

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Proof.

By refining the proof of Czelakowski-Dziobiak theorem. □

Proof of Pigozzi theorem, continued

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Let $\mathcal{F}$ be a finite family of finite algebras and $\mathcal{R} = \mathbf{SP}(\mathcal{F})$. Then $\mathcal{R}_{SI} \subseteq \mathbf{S}(\mathcal{F})$ is a finite family of finite algebras and hence it is strictly elementary. Now use previous facts. □

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Let $\mathcal{F}$ be a finite family of finite algebras and $\mathcal{R} = \mathbf{SP}(\mathcal{F})$. Then $\mathcal{R}_{SI} \subseteq \mathbf{S}(\mathcal{F})$ is a finite family of finite algebras and hence it is strictly elementary. Now use previous facts. □

Problem

Let $\mathcal{R}$ be a relatively congruence-distributive quasivariety and assume that the class $\mathcal{R}_{SI}$ of $\mathcal{R}$-subdirectly irreducible algebras is strictly elementary. Must $\mathcal{R}$ be finitely based?

The End

Thank you for your attention :-)